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ON SOME VARIATIONAL PRINCIPLES IN MECHANICS OF CONTINUOUS MEDIA

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Some problems of the analytical mechanics of continuous media are considered. The kinematc constraints, restricting the motion of elements of a continuous medium are separated into internal and external constraints; the internal surface stresses in a continuous medium are treated as reactive forces of the internal constraints. Since the work of these latter on possible displacements of elements of the continuous medium is not zero in the general case, then the internal constraints in a continuous medium should be referred to the category of nonideal constraints. The general equation of the dynamics of a continuous medium expressing the d'Alembert-Lagrange variational principle and including all the dynamic laws is examined. The work of the internal surface stresses in this equation can be given by using the first and second laws of thermodynamics, whereupon the general equation of the dynamics of continuous media can be represented in two other forms. Furthermore, an extension of the Gauss and Chetaev principles to continuous media is given.

1. Let us consider the motion of some continuous medium relative to an inertial rectangular Cartesian coordinate system $O_1x_1x_2x_3$. Let the continuous medium occupy a finite domain D of the space $x_1x_2x_3$, bounded by a closed surface Σ; in the general case D and Σ vary with time t. Let $\mathbf{r}$ denote a radius-vector relative to the origin O_1 of some point of the dimain or its boundary, and let $\rho = \rho(\mathbf{r}, t)$, $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ and $\mathbf{w} = d\mathbf{v}/dt = \mathbf{w}(\mathbf{r}, t)$ be the density and velocity and acceleration fields of the continuous medium at the time t. In the domain of continuous motions described by smooth functions, the general equations of motion of a continuous medium are [1]

$$\rho\mathbf{w} = \rho\mathbf{F} + \frac{\partial \mathbf{p}_1}{\partial x_1} + \frac{\partial \mathbf{p}_2}{\partial x_2} + \frac{\partial \mathbf{p}_3}{\partial x_3} \tag{1.1}$$

$$\frac{1}{\rho}\frac{d\rho}{dt} + \operatorname{div}\mathbf{v} = 0 \tag{1.2}$$

As is known, these equations express Newton's second law and the law of mass conservation. Here $\mathbf{F}$ denotes the density of the mass forces, and $\mathbf{p}_i$ is the density of the internal surface stresses on areas orthogonal to the axes x_i.

Ordinarily (1.1) are derived from the integral relation [1, 2]

$$\frac{d}{dt}\int_\tau \rho\mathbf{v}\,d\tau = \int_\tau \rho\mathbf{F}\,d\tau + \int_\sigma \mathbf{p}_n\,d\sigma \tag{1.3}$$

expressing the theorem of momentum in application to an arbitrary volume τ imagined extracted from within the domain D which consists of the same particles of the medium and is bounded by a closed surface σ. Under the condition that the components of the stress tensor $\mathbf{P}$ are smooth functions within the volume τ, the Gauss-Ostrogradskii formula

$$\int_\sigma \mathbf{p}_n \, d\sigma = \int_\tau \left(\frac{\partial \mathbf{p}_1}{\partial x_1} + \frac{\partial \mathbf{p}_2}{\partial x_2} + \frac{\partial \mathbf{p}_3}{\partial x_3} \right) d\tau$$

is valid, from which it follows that the effects governed by the surface density of the stresses $\mathbf{p}_n$ on the boundary σ are equivalent to the effects governed by the volume density within τ [2]

$$\mathbf{C}(\mathbf{r}) = \frac{\partial \mathbf{p}_1}{\partial x_1} + \frac{\partial \mathbf{p}_2}{\partial x_2} + \frac{\partial \mathbf{p}_3}{\partial x_3} \tag{1.4}$$

Evidently any particle of the continuous medium is not completely free since the other particles surrounding it do not permit it to move arbitrarily in space. For each particle, the particles surrounding it impose certain constraints on the displacement (or velocity) which can be expressed as sufficiently general conditions for conservation of the continuity of the medium and the continuity of the displacement (velocity) field of the medium particles. In substance, such kinematic-type constraints are, independently of the forces acting on the medium and on its motion laws, internal constraints imposed on all the adjacent elements of the medium [3]; the constraint equations can be written as conditions for the sufficient smoothness of the displacements (velocities).

Surface forces originate within a continuous medium because of the effect of adjacent surrounding particles on the particle of the medium, i.e. they are interaction forces between the particles which originate because of the presence of surface cohesions [2] between adjacent elements of the continuous medium. Hence, surface stresses with density $\mathbf{p}_n$, or their equivalent effects with the mass density (1.4) can naturally be treated as reactive forces of the internal constraints of the continuous medium.

Let us note that generally in the mechanics of continuous medium, besides the internal constraints, there are also the external constraints, namely, diverse kinetic boundary conditions, which can be given on the boundary Σ.

In conformity with the definition accepted in analytical mechanics, we shall understand the possible displacements $\delta\mathbf{r}$ of particles of a continuous medium to be elementary displacements admitted at a given time by the constraints imposed on the system. Then proceeding from the definition of the constraints accepted above, we conclude that the possible displacements are arbitrary smooth functions of the locations of points of the domain D without violating the continuity of the medium, and therefore, satisfying conditions for points of the boundary Σ which result from the kinematic boundary conditions.

Assuming that the possible displacements $\delta\mathbf{r}$ have partial derivatives with respect to x_i which are integrable in the domain D, let us compute the work of the reactive force (1.4) of the internal constraints on the possible system displacement

$$\int_D \left(\frac{\partial \mathbf{p}_1}{\partial x_1} + \frac{\partial \mathbf{p}_2}{\partial x_2} + \frac{\partial \mathbf{p}_3}{\partial x_3} \right) \cdot \delta\mathbf{r} \, d\tau = \int_\Sigma \mathbf{p}_n \cdot \delta\mathbf{r} \, d\sigma + \delta A^i \tag{1.5}$$

where the work of the internal surface forces on the possible displacement, by definition, is

$$\delta A^i = -\int_D \left(\mathbf{p}_1 \cdot \frac{\partial \delta\mathbf{r}}{\partial x_1} + \mathbf{p}_2 \cdot \frac{\partial \delta\mathbf{r}}{\partial x_2} + \mathbf{p}_3 \cdot \frac{\partial \delta\mathbf{r}}{\partial x_3} \right) d\tau \tag{1.6}$$

The equality (1.5) indicates that the work of a force with mass density (1.4) equals the sum of the work of the external and internal surface forces.

Furthermore, let us assume for simplicity that the stress tensor is symmetric, i.e. its components satisfy the conditions $p_{ij} = p_{ji}$. Under these conditions, the work of the internal surface forces is [1, 2]

$$\delta A^i = -\int\limits_D p_{ij}\delta\varepsilon_{ij}\, d\tau = -\int\limits_D \frac{p_{ij}}{\rho}\delta\varepsilon_{ij}\, dm \tag{1.7}$$

where

$$\delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\delta x_i}{\partial x_j} + \frac{\partial\delta x_j}{\partial x_i}\right), \qquad dm = \rho d\tau \tag{1.8}$$

Let us note that if the possible displacements of the medium particles are displacements of a solid, then $\delta\varepsilon_{ij} = 0$ and $\delta A^i = 0$.

In the general case it follows from (1.7) that the work of the internal surface forces on the possible displacements is $\delta A^i \neq 0$. This means that if the stresses in a continuous medium are considered as the reaction of the internal constraints, then these latter should generally be referred to the category of nonideal constraints, or constraints with friction [4, 5]. Hence, in order to determine the motion and stresses of a continuous medium, some additional relationships which play the same part as does the law of friction in the mechanics of systems with a finite number of degrees of freedom with nonideal constraints, must be given in addition to (1.1) and (1.2). As is known, the stresses in a continuous medium are closely connected with its deformations; The stress-strain relationships are generally the relationships which should be specified. The specific connection between the stresses and strains is determined by selecting some model of a continuous medium [1].

If it turns out that $\delta A^i = 0$ for all possible displacements, then in this case the internal constraints are referred to the category of deal constraints. An ideal incompressible fluid for which the stress tensor components are $p_{ij} = -p\delta_{ij}$, where p is the hydrodynamic pressure, δ_{ij} is the Kronecker symbol and the density ρ of each particle is constant, is an illustration of a modèl of a continuous medium with ideal constraints. Under these conditions, we obtain from (1.7)

$$\delta A^i = \int\limits_D \frac{p}{\rho}\delta_{ij}\delta\varepsilon_{ij}\, dm = \int\limits_D \frac{p}{\rho}\delta\varepsilon_{ii}\, dm = 0,$$

since for an incompressible fluid in conformity with (1.2)

$$\delta\varepsilon_{ii} = div\delta\mathbf{r} = 0$$

Let us note that in the general case for an ideal compressible fluid

$$\delta A^i = \int\limits_D p\delta\frac{1}{\rho}\, dm \neq 0$$

Let us multiply (1.1) scalarly by $\delta\mathbf{r}d\tau$ and let us integrate over the domain D or any part of it τ. We hence obtain the equation

$$\int\limits_D \rho(\mathbf{F} - \mathbf{w})\cdot\delta\mathbf{r}\, d\tau + \int\limits_\Sigma \mathbf{p}_n\cdot\delta\mathbf{r}\, d\sigma + \delta A^i = 0 \tag{1.9}$$

in which the work of the internal surface forces is expressed by (1.6) or (1.7). Taking account of (1.4) and (1.5), Eq. (1.9) can also be written in the equivalent form

$$\int_D [\rho(\mathbf{F}-\mathbf{w})+\mathbf{C}(\mathbf{r})]\cdot\delta\mathbf{r}\,d\tau=0 \tag{1.10}$$

Equation (1.9) has been obtained by using (1.1). Conversely, (1.9) (or (1.10)) can be taken as the initial equation and (1.1) derived therefrom. Moreover, all the general theorems of the dynamics of a continuous medium can be obtained from (1.9) by applying it to an arbitrary volume τ within the domain D consisting of the same particles of the medium and taking account of (1.2). Indeed, among the possible particle displacements we consider the translational displacements of the volume τ as a solid by setting $\delta\mathbf{r}=\delta\mathbf{r}_0$; hence $\delta A^i=0$. Taking the vector $\delta\mathbf{r}_0$ outside the integral sign in (1.9) and dividing by it, we obtain (1.3) which expresses the momentum theorem.

Now, let us consider rotational displacements of the volume τ as a solid around the point O_1 among the possible displacements by setting $\delta\mathbf{r}=\delta\boldsymbol{\varphi}\times\mathbf{r}$. Taking into account that hence $\delta A^i=0$, we obtain the theorem of angular momentum

$$\frac{d}{dt}\int_\tau \mathbf{r}\times\rho\mathbf{v}\,d\tau=\int_\tau \mathbf{r}\times\rho\mathbf{F}\,d\tau+\int_\sigma \mathbf{r}\times\mathbf{p}_n\,d\sigma \tag{1.11}$$

by applying the formula $\mathbf{a}\cdot(\delta\boldsymbol{\varphi}\times\mathbf{r})=\delta\boldsymbol{\varphi}\cdot(\mathbf{r}\times\mathbf{a})$, taking the vector $\delta\boldsymbol{\varphi}$ outside the integral sign and dividing by it.

Finally, let the possible displacements $\delta\mathbf{r}$ coincide with the actual diaplacements in time dt. Setting $\delta\mathbf{r}=d\mathbf{r}=\mathbf{v}dt$, we obtain the theorem about the kinetic energy from (1.9)

$$dE=\int_\tau \rho\mathbf{F}\cdot d\mathbf{r}\,d\tau+\int_\sigma \mathbf{p}_n\cdot d\mathbf{r}\,d\tau+dA^i,\qquad E=\frac{1}{2}\int_\tau \rho\mathbf{v}^2\,d\tau \tag{1.12}$$

Here dA^i denotes the work of the internal surface forces on the actual displacement defined by the equalities (1.6) or (1.7) if δ is replaced in the latter by d.

Let us note that the actual displacements are among the possible displacements only in the case of stationary constraints. The internal constraints in a continuous medium are stationary. As regards the external constraints, they can also be nonstationary. In the presence of nonstationary constraints, they can be imagined discarded by replacing them by reactions which are surface stresses $\mathbf{p}_n$ not known in advance, and by setting $\delta\mathbf{r}=d\mathbf{r}$, Eq. (1.12) can be obtained on whose right side the work of the external surface stresses $\mathbf{p}_n$, the effect of external constraints on a continuous medium, will enter as in the case of external stationary nonideal constraints.

Therefore, (1.9) includes all the dynamic laws, whereupon it is the general equation of the dynamics of continuous media with a symmetric stress tensor.

Ordinarily one starts from (1.3) and (1.11) in the dynamics of continuous media and then obtains the theorem (1.12) by using (1.1) and (1.2). It is however, more logical to proceed from (1.9) which contains all the dynamics of continuous media. It should also be kept in mind that (1.9), in contrast to (1.1), and the resulting equations (1.3) and (1.11) are true for any motions, including even discontinuous motions under the conditions that the integrals in these equations are finite [1].

Let us note that an equation of the form (1.9) or (1.10) is presented in the book [2] as a theorem of possible powers.

It should be stresses that in contrast to the exposition in [3], the work δA^i of the internal surface stresses as well as the work of the external surface stresses $\mathbf{p}_n$ among which can also be the reactions of the external constraints, all are included in the general equation (1.9) of the dynamics of continuous media. The presence of terms expressing the work of the constraint reactions on the possible displacements in the general equation of dynamics is generally characteristic for systems with nonideal constraints with both, a finite and an infinite number of degrees of freedom. Because the constraints are nonideal knowledge of the active forces applied to the system is inadequate to the determination of its motion, as well as the constraint reaction, in contrast to systems with ideal constraints: at least the work of the constraint reaction on the possible displacements should be given in addition to the active forces.

In the case of the continuous media under consideration, the work of the internal surface forces can be found from the heat influx equation

$$\delta u = \frac{p_{ij}}{\rho}\delta\varepsilon_{ij} + \delta q^{(e)} + \delta q^{xx} \tag{1.13}$$

if the change δu in the internal energy and the elementary external heat influxes $\delta q^{(e)}$ and others nonthermal kinds of energy δq^{xx} computed per unit mass different from the work of the macroscopic mechanical forces are known. As is known [1], Eq. (1.13) is equivalent to the first law of thermodynamics, the energy conservation law, in the presence of the dynamic equations (1.12). Taking (1.7) and (1.13) into account, the general equation of dynamics (1.9) can be represented as

$$\int_D \rho\,[(\mathbf{F} - \mathbf{w})\cdot\delta\mathbf{r} - \delta u + \delta q^{(e)} + \delta q^{xx}]\,d\tau + \int_\Sigma \mathbf{p}_n\cdot\delta\mathbf{r}\,d\sigma = 0 \tag{1.14}$$

if it is assumed that the internal system energy possesses the additivity property. If $\mathbf{F}$, u, $q^{(e)}$ and q^{xx} are known at each point within the domain D and $\mathbf{p}_n$ on the surface Σ, then (1.14) permits the determination of the motion of the continuous medium under given initial and boundary conditions as well as the internal stresses.

According to the second law of thermodynamics, the elementary external heat influx is related to the change in entropy δs and the uncompensated heat $\delta q' \geqslant 0$ by the relationship

$$T\delta s = \delta q^{(e)} + \delta q' \tag{1.15}$$

where T is the absolute temperature. Taking this relationship into account, the general equation of dynamics can also be represented as

$$\int_D \rho\,[(\mathbf{F} - \mathbf{w})\cdot\delta\mathbf{r} - \delta u + T\delta s - \delta q' + \delta q^{xx}]\,d\tau + \int_\Sigma \mathbf{p}_n\cdot\delta\mathbf{r}\,d\tau = 0 \tag{1.16}$$

A closed system of equations of motion and state of a continuous medium [6] can be obtained from (1.16) for given F, u, q' and q^{xx}. In this sense, (1.14) or (1.16) play the same part in the mechanics of continuous media as the general equation of dynamics of systems with a finite number of degrees of freedom with ideal constraints.

Equation (1.9) expresses the d'Alembert-Lagrange variational principle in the dynamics of continuous media. Infinitesimal displacements from the actual given motion of the systems are considered therein. However, this principle is not related to the extremum of some functional but it can be modified so that it will express the extremum of some expression.

Let us examine two such modifications of the d'Alembert-Lagrange principle which extend the Gauss principle [7] and the Chetaev principle [8, 9] to a continuous medium.

2. First let us extend the Chetaev theorem and the Gauss principle to continuous media.

Let us consider, besides the real motions of the continuous medium, the motions of the medium particles, conceivable according to Gauss [7], which have the very same values of the radius vectors r and the velocities v at the time t as in the real motion, and satisfy the conditions imposed by the constraints. In other words, at time t the conceivable motions differ from the real motion only by the accelrations which should satisfy the conditions of concervation of the continuity and indissolubility (1.2) and the kinematic boundary conditions.

Let us consider some fictitious motion of the medium in an infinitesimal time interval from t to $t + dt$. The possible particle displacements $\delta \mathbf{r}$ are hence related to the changes in acceleration

$$\Delta \mathbf{w} = \delta \mathbf{v} / dt - \mathbf{w} \tag{2.1}$$

by the relationships [8]

$$\delta \mathbf{r} = \Delta \mathbf{w} \frac{(dt)^2}{2} \tag{2.2}$$

Here $\delta \mathbf{v}$ is the change in the velocity in the fictitious motion during the time dt. Taking (2.2) into account, (19) becomes

$$\int_D \rho (\mathbf{F} - \mathbf{w}) \cdot \Delta \mathbf{w} \, d\tau + \int_\Sigma \mathbf{p}_n \cdot \Delta \mathbf{w} \, d\sigma + \frac{2}{(dt)^2} \delta A^i = 0 \tag{2.3}$$

where δA^i is defined by (1.6) taking into account (2.2).

Let us (imaginarily) free the continuous medium from part (or from all) of the constraints imposed on it, from the external kinematic constraints, say. The possible displacements of the initial system are among the possible displacements of the liberated system, hence the following equation is valid:

$$\int_D \rho \left(\mathbf{F} - \frac{\partial \mathbf{v}}{dt}\right) \cdot \Delta \mathbf{w} \, d\tau + \int_\Sigma \mathbf{p}_n^\partial \cdot \Delta \mathbf{w} \, d\sigma + \frac{2}{(dt)^2} \partial A^i = 0 \tag{2.4}$$

Here $\partial \mathbf{v}$ is the change in velocity in the real liberated motion during the time dt, $\mathbf{p}_n^\partial$ are surface stresses in the system freed from part of the constraints, and ∂A^i is defined by a formula such as (1.6) with $\mathbf{p}_i$ replaced by $\mathbf{p}_i^\partial$ and taking (2.2) into account. Subtracting (2.4) from (2.3), we obtain

$$\int_D \rho \left(\frac{\partial \mathbf{v}}{dt} - \mathbf{w}\right) \cdot \Delta \mathbf{w} \, d\tau + \int_\Sigma (\mathbf{p}_n - \mathbf{p}_n^\partial) \cdot \Delta \mathbf{w} \, d\sigma + \frac{2}{(dt)^2} (\delta A^i - \partial A^i) = 0$$

This equation can be rewritten as

$$A_{d\delta} + A_{d\partial} - A_{\partial\delta} + \int_\Sigma (\mathbf{p}_n - \mathbf{p}_n^\partial) \cdot \Delta \mathbf{w} \, d\sigma + \frac{2}{(dt)^2} (\delta A^i - \partial A^i) = 0$$

where the quantities

$$A_{d\delta} = \frac{1}{2} \int_D \rho \left(\mathbf{w} - \frac{\delta \mathbf{v}}{dt}\right)^2 d\tau, \qquad A_{d\partial} = \frac{1}{2} \int_D \rho \left(\mathbf{w} - \frac{\partial \mathbf{v}}{dt}\right)^2 d\tau$$

$$A_{\partial\delta} = \frac{1}{2} \int_D \rho \left(\frac{\partial \mathbf{v}}{dt} - \frac{\delta \mathbf{v}}{dt}\right)^2 d\tau$$

characterize the measure of the deviations between the real (d), the actual freed (∂), and the imaginary (δ) motions of the medium during the time dt. Taking account of (1.6) and (2.2), it is seen that the expression

$$\frac{2}{(dt)^2}(\delta A^i - \partial A^i) = -\int_D (\mathbf{p}_i - \mathbf{p}_i^{\partial})\cdot\frac{\partial \Delta \mathbf{w}}{\partial x_i}\,d\tau$$

is the difference between the work of the internal surface stresses in the actual and freed motions of the medium of the "displacements" $\Delta\mathbf{w}$. Since the quantities $A_{d\partial}$ and $A_{d\delta}$ are essentially positive quantities when the motions (∂) and (δ) do not coincide with the motion (d) we obtain the two following equations from (2.5)

$$A_{d\delta} + \int_\Sigma (\mathbf{p}_n - \mathbf{p}_n^{\partial})\cdot\Delta\mathbf{w}\,d\sigma - \int_D (\mathbf{p}_i - \mathbf{p}_i^{\partial})\cdot\frac{\partial\Delta\mathbf{w}}{\partial x_i}\,d\tau < A_{\partial\delta} \qquad (2.6)$$

$$A_{d\partial} + \int_\Sigma (\mathbf{p}_n - \mathbf{p}_n^{\partial})\cdot\Delta\mathbf{w}\,d\sigma - \int_D (\mathbf{p}_i - \mathbf{p}_i^{\partial})\cdot\frac{\partial\Delta\mathbf{w}}{\partial x_i}\,d\tau < A_{\partial\delta}$$

which expresses the following Theorem :

A measure of the deviation of the actual (d) motion of a continuous medium from some fictitious (δ) (actual freed (∂)) motion, increased by the difference between the work of the external and internal surface stresses in the actual and freed motions on the displacement (2.1) is less than the measure of the deviation of the fictitious (δ) motion from the actual freed (∂) motion.

This Theorem is the extension of the Chetaev theorem [8] for systems with ideal constraints to a continuous medium.

Now, let the particles of the medium be freed of all constraints, both external and internal, at the time t, i.e. let them become perfectly free. Their accelerations $\partial\mathbf{v}/dt$ will hence equal $\mathbf{F}$, all the stresses will vanish $\mathbf{p}_i^{\partial} = 0$, and the second equation in (2.6) becomes

$$A_{d\partial} + \int_\Sigma \mathbf{p}_n\cdot\Delta\mathbf{w}\,d\sigma - \int_D \mathbf{p}_i\cdot\frac{\partial\Delta\mathbf{w}}{\partial x_i}\,d\tau < A_{\partial\delta} \qquad (2.7)$$

where

$$A_{d\partial} = \frac{1}{2}\int_D \rho(\mathbf{w} - \mathbf{F})^2\,d\tau, \qquad A_{\partial\delta} = \frac{1}{2}\int_D \rho\left(\mathbf{F} - \frac{\delta\mathbf{v}}{dt}\right)^2 d\tau \qquad (2.8)$$

The inequality (2.7) has been obtained as a corollary of the second inequality in (2.6). However, it can be established independently of (2.6) . Indeed, let us transpose the term $A_{\partial\delta}$ to the left side of the inequality, let us substitute (2.8), let us reduce similar terms, and let us perform the evident manipulation, whereupon we hence obtain the inequality

$$\int_D \left[\rho(\mathbf{F} - \mathbf{w})\cdot\Delta\mathbf{w} - \mathbf{p}_i\cdot\frac{\partial\Delta\mathbf{w}}{\partial x_i}\right]d\tau + \int_\Sigma \mathbf{p}_n\cdot\Delta\mathbf{w}\,d\sigma < \frac{1}{2}\int_D \rho\left(\mathbf{w} - \frac{\delta\mathbf{v}}{dt}\right)^2 d\tau \qquad (2.9)$$

Using (1.1) and the Gauss-Ostrogradskii formula, we see that the left side of (2.9) is zero, which indeed proves the latter's validity. Taking account of (2.1), let us rewrite (2.7) as

$$A_{d\partial} - \int_\Sigma \mathbf{p}_n\cdot\mathbf{w}\,d\sigma + \int_D \mathbf{p}_i\cdot\frac{\partial\mathbf{w}}{\partial x_i}\,d\tau < A_{\partial\delta} - \int_\Sigma \mathbf{p}_n\cdot\frac{\delta\mathbf{v}}{dt}\,d\sigma + \int_D \mathbf{p}_i\cdot\frac{\partial}{\partial x_i}\frac{\delta\mathbf{v}}{dt}\,d\tau \qquad (2.10)$$

in which the quantities $A_{d\partial}$ and $A_{\partial\delta}$ are defined by (2.8). The inequality (2.10) is the extension of the Gauss principle to continuous media.

The measure of the deviation of the actual motion of a continuous medium from the motion of free particles, diminished by the work of the external and internal stresses on the actual accelerations, is less than the measure of the deviation of any fictitious motion from the free particle motion diminished by the work of the external and internal stresses on the accelerations of the fictitious motion.

This Theorem extends the Gauss principle [5] to continuous media for systems with a finite number of degrees of freedom constrained by nonideal constraints. This principle can be formulated somewhat differently. Indeed, let us substitute (2.8) into (2.10) and let us reduce similar terms. We consequently obtain the inequality

$$\frac{1}{2}\int_D \rho\,(\mathbf{w}^2 - 2\mathbf{F}\cdot\mathbf{w})\,d\tau - \int_\Sigma \mathbf{p}_n\cdot\mathbf{w}\,d\sigma + \int_D \mathbf{p}_i\cdot\frac{\partial\mathbf{w}}{\partial x_i}\,d\tau < R \tag{2.11}$$

$$R = \frac{1}{2}\int_D \rho\left[\left(\frac{\delta\mathbf{v}}{dt}\right)^2 - 2\mathbf{F}\cdot\frac{\delta\mathbf{v}}{dt}\right]d\tau - \int_\Sigma \mathbf{p}_n\cdot\frac{\delta\mathbf{v}}{dt}\,d\sigma + \int_D \mathbf{p}_i\cdot\frac{\partial}{\partial x_i}\frac{\delta\mathbf{v}}{dt}\,d\tau \tag{2.12}$$

expressing the Theorem :

Among all the particle accelerations $\delta\mathbf{v}/dt$ of a continuous medium at each instant which are satisfying the constraints, the actual accelerations will be such accelerations $\mathbf{w}$ as will minimize the functional (2.12).

Let us note that the quantity R equals the difference between the acceleration energies of the continuous medium in the fictitious motion

$$S = \frac{1}{2}\int_D \rho\left(\frac{\delta\mathbf{v}}{dt}\right)^2 d\tau$$

and the work of all the mass and surface forces

$$\int_D \rho\mathbf{F}\cdot\frac{\delta\mathbf{v}}{dt}\,d\tau + \int_\Sigma \mathbf{p}_n\cdot\frac{\delta\mathbf{v}}{dt}\,d\sigma - \int_D \mathbf{p}_i\cdot\frac{\partial}{\partial x_i}\frac{\delta\mathbf{v}}{dt}\,d\tau$$

on the accelerations of the fictitious motion.

Equilibrium is a particular case of motion; it holds when all the medium particles are at rest, i.e. when $\mathbf{v} = \mathbf{w} = 0$. Therefore, among all the kinematically admissible [2] displacements $\mathbf{u}'$ satisfying the constraints, the actual displacements $\mathbf{u}$ minimize the functional

$$V = \int_D \mathbf{p}_i\cdot\frac{\partial\mathbf{u}'}{\partial x_i}\,d\tau - \int_D \rho\mathbf{F}\cdot\mathbf{u}'\,d\tau - \int_\Sigma \mathbf{p}_n\cdot\mathbf{u}'\,d\sigma$$

The first theorem of potential energy [2] follows from this Theorem as a particular case.

Note. The inequality (2.11) is close to the inequality (2.2) in [10], (but does not agree completely with it).

3. An interesting modification can be given to the Gauss principle for a continuous medium.

Let us consider some fictitious Gaussian motion of a continuous medium during the time from t to $t + dt$. During this time the medium particle will perform an infinitesimal displacement in the fictitious motion

$$\mathbf{r}(t + dt) - \mathbf{r}(t) = \left(\mathbf{v} + \frac{1}{2}\delta\mathbf{v}\right)dt + \ldots$$

where $\delta\mathbf{v}$ is the change in velocity in the fictitious motion during the time dt. The expression for the work of the mass and surface forces on an infinitesimal displacement of the fictitious motion is

$$\int_D \rho \mathbf{F} \cdot \left(\mathbf{v} + \frac{1}{2}\delta\mathbf{v}\right) dt\, d\tau + \int_\Sigma \mathbf{p}_n \cdot \left(\mathbf{v} + \frac{1}{2}\delta\mathbf{v}\right) dt\, d\sigma + \delta' A^i + \ldots$$

where $\delta' A^i$ is defined by (1.6) with $\delta\mathbf{r}$ replaced by $(\mathbf{v} + {}^1/_2\delta\mathbf{v})\ dt$. Let us subtract the expression for the work on the considered infinitesimal displacement of the forces $\rho\mathbf{w}$ which would be sufficient to produce the actual motion if the medium particles were free

$$\int_D \rho \mathbf{w} \cdot \left(\mathbf{v} + \frac{1}{2}\delta\mathbf{v}\right) dt\, d\tau + \ldots$$

We consequently obtain the expression

$$A_\mu = \int_D \rho(\mathbf{F} - \mathbf{w}) \cdot \left(\mathbf{v} + \frac{1}{2}\delta\mathbf{v}\right) dt\, d\tau + \int_\Sigma \mathbf{p}_n \cdot \left(\mathbf{v} + \frac{1}{2}\delta\mathbf{v}\right) dt\, d\sigma + \delta' A^i + \ldots \tag{3.1}$$

of the work in an elementary cycle consisting of the direct fictitious motion in a field of given forces and the back (reverse) motion in the field of forces which would be sufficient to produce the actual motion if all the medium particles were perfectly free.

For an analogous cycle constructed for the actual medium motion, the work is

$$A = \int_D \rho(\mathbf{F} - \mathbf{w}) \cdot \left(\mathbf{v} + \frac{1}{2}d\mathbf{v}\right) dt\, d\tau + \int_\Sigma \mathbf{p}_n \cdot \left(\mathbf{v} + \frac{1}{2}d\mathbf{v}\right) dt\, d\sigma + d' A^i + \ldots \tag{3.2}$$

where $d' A^i$ is defined by (1.6) with $\delta\mathbf{r}$ replaced by $(\mathbf{v} + {}^1/_2 d\mathbf{v})\ dt$.

Subtracting (3.2) from (3.1), we obtain

$$A_\mu - A = \int_D \rho(\mathbf{F} - \mathbf{w}) \cdot (\delta\mathbf{v} - d\mathbf{v}) \frac{dt}{2} d\tau + \int_\Sigma \mathbf{p}_n \cdot (\delta\mathbf{v} - d\mathbf{v}) \frac{dt}{2} d\sigma + \delta' A^i - d' A^i$$

Let Δ denote the change during passage from the actual motion to a slightly different fictituous motion. Taking account of (2.1) this expression can be represented as

$$\Delta A = \int_D \rho(\mathbf{F} - \mathbf{w}) \cdot \Delta\mathbf{w} \frac{(dt)^2}{2} d\tau + \int_\Sigma \mathbf{p}_n \cdot \Delta\mathbf{w} \frac{(dt)^2}{2} d\sigma + \Delta\delta' A^i \tag{3.3}$$

$$\Delta\delta' A^i = -\int_D \mathbf{p}_i \cdot \frac{\partial \Delta\mathbf{w}}{\partial x_i} \frac{(dt)^2}{2} d\tau$$

Taking (2.2) into account, we find from (1.9) that

$$\Delta A = 0$$

Applying the operation Δ once again to (3.3) and taking into account that $\Delta\mathbf{F} = \Delta\mathbf{p}_i = 0$ for forces and stresses independent of accelerations, we obtain

$$\Delta^2 A = -\frac{(dt)^2}{2}\int_D \rho\,(\Delta \mathrm{w})^2\,d\tau < 0$$

Therefore, it has been established for a continuous medium [8, 9].

Chetaev principle. The work of an elementary cycle consisting of the forward motion of a continuous medium in the field of mass and surface forces and the reverse motion in a field of forces which would be sufficient to produce the actual motion if the medium particles were perfectly free, has (at least a relative) maximum in the class of fictitous Gaussian motions for the actual motion.

Just as the D'Alembert-Lagrange principle, this principle can also be expressed taking into account the first and second laws of thermodynamics.

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METHOD OF SUCCESSIVE APPROXIMATIONS FOR THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER PROBLEMS (LOCALLY SELF-SIMILAR CASE)

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We present an analytical method for the computation of problems of incompressible boundary layer theory based on an application of the method of successive approximations. The system of equations is reduced to a form suitable for integration. Parameters characterizing the external flow and the body geometry are contained only in the coefficients of the system and do not enter into the boundary conditions. The transformed momentum equations are inte-